MATHEMATICS TEXTBOOK ANALYSIS: A GUIDE TO CHOOSING THE APPROPRIATE MATHEMATICS TEXTBOOK

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There is a lot of research which has been carried out on the influence of mathematical textbooks on student teaching and learning of mathematics (Morgan, 1995; 2005; Dowling, 1996). Since the mathematics textbook has such a pivotal role within the classroom, the impact of the mathematics textbook on student learning is undeniable. This research, by means of mathematics textbook analysis, investigates the effectiveness of the Pearson mathematics textbooks for developing student comprehension and motivation. In order to analyse the textbooks, we have used two theoretical frameworks to analyse Pearson mathematics textbooks. The theoretical frameworks used are Kilpatrick et al's (2001) five strands of mathematics proficiency and Marton et al's (2004) Variation theory. From the two frameworks, we developed an analytical framework, which we used to analyse the textbooks. Conclusions from the textbook analysis are that the textbook used in the classroom is very important in determining how learners view mathematics, the textbook helps in enabling or restricting learners to communicate their mathematics ideas and also the textbook should probe learners for higher mathematical reasoning by asking questions that require them to give more explanation and good justifications for their responses in the mathematics classroom.

INTRODUCTION

Mathematics is one of humanity's great achievements (Mckenzie, 2001). In order for people to participate fully in society, they must know basic mathematics. Innumeracy deprives them not only of opportunity but also of competence in everyday tasks, for example banking and shopping. Citizens who cannot reason mathematically are cut off from whole realms of human endeavour (Kilpatrick et al., 2001). Mathematics is an intellectual achievement of great sophistication and beauty that uses the power of deductive reasoning (Muller & Maher, 2009). Deductive reasoning is a logical process in which a conclusion is based on the accordance of multiple premises that are generally assumed to be true.

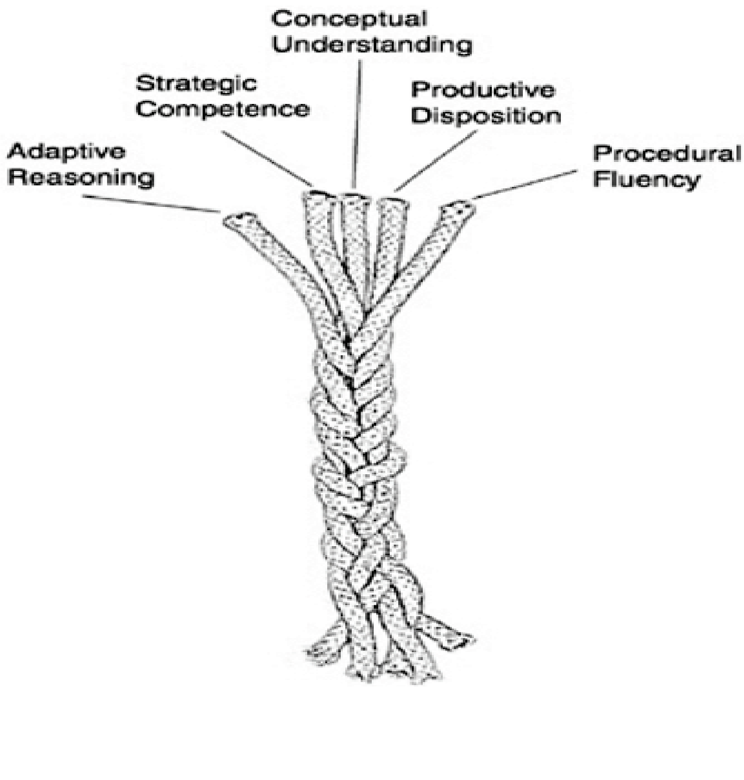
The mathematics students need to learn today is not the same mathematics that their parents and grandparents needed to learn (Bauer, 2013). When today's students become adults, they will face new demands for mathematics proficiency that school mathematics needs to attempt to anticipate. The mathematics textbook which students use should look at all aspects in an attempt to anticipate the needs of the students such as introducing deductive reasoning explicitly so that learn how to use the reasoning in mathematics (Muller & Maher, 2009).

According to the Curriculum, there are four cognitive levels in mathematics, namely: Knowledge, Routine Procedures, Complex Procedures and Problem solving. In this study, we have used two theoretical frameworks to analyse Pearson mathematics textbooks. The theoretical frameworks used are Kilpatrick et al's (2001) five strands of mathematics proficiency and Marton et al's (2004) Variation theory.

KILPATRICK ET AL'S FIVE STRANDS

Kilpatrick, Swafford and Findell (2001) conducted research in mathematics classrooms in the US, over a long period of time and they came up with a theory that in order for a learner to successfully learn mathematics, five strands of mathematics proficiency should be developed: Conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Procedural fluency refers to the skill in carrying out procedures flexibly, accurately and appropriately (Kilpatrick et al., 2001). Strategic competence refers to the ability to formulate, represent, and solve mathematical problems while Adaptive reasoning refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick et al., 2001). Productive disposition refers to the ability to 'see' mathematics as sensible, useful, and worthwhile (Kilpatrick et al, 2001). Conceptual understanding enables learners to learn new ideas by connecting those ideas to what they already know. It supports retention and also prevents common errors (Kilpatrick et al., 2001). Knowledge learned with understanding provides a foundation for generating new knowledge and for solving unfamiliar problems (Bransford et al., 1999).

The five strands are interwoven as shown in the diagram below and hence development of one strand depends on development of the other strands (Kilpatrick et al., 2001: 116).



Kilpatrick et al's (2001) theory argues for learning with understanding because it is more powerful than simply memorizing and also organisation improves retention, promotes fluency and facilitates learning related material. Having a deep understanding requires that learners connect pieces of knowledge, and that connection in turn is a key factor in whether they can use what they know productively in solving problems. Kilpatrick et al (2001) argue that their theory can be adapted to any mathematics teaching and learning environment.

VARIATION THEORY

Variation theory is a theory about learning (Runesson, 2005). Learning is defined as the "process of becoming capable of doing something as a result of having had certain experiences" (Marton, Runesson and Tsui, 2004: 5). "Learning always has an object" (Runesson, 2006: 406) because there is always something that has to be learnt, and once the critical features of what is being learnt are discerned, then learning has taken place (Runesson, 2006). The object is defined as "that which is the focus of attention" (Watson and Mason, 2006: 100). Marton et al. (2004) and Runesson (2005/ 2006) give a detailed discussion about variation theory. Variation theory focuses on the differences in the dimension or values of a feature. According to variation theory, it is not possible to discern a certain way of thinking about something without the contrast of other ways of thinking about the same thing (Marton et al. 2004). It is important to note that Marton et al. (2004) say that they are not advocating for an overall application of variation theory but they believe that variation theory can be used to enable learners to experience the features that are critical for particular learning and also to help learners in developing certain capabilities. Marton et al. (2004) define capabilities as the object of learning. The object of learning is defined by its critical features and the critical features are those features that should be discerned in order for the meaning that is aimed for to be understood. Mathematics as a subject has objects of learning, for example, algebra as a topic is referred to as an object of learning. The object of learning has the *general* and the *specific* component. The general component involves the nature of the capability, for example, the general component of algebra involves remembering, interpreting, discerning and grasping the various features within algebra. The specific component involves the actual 'thing' to which the acts of remembering, discerning and grasping are carried out (Marton et al, 2004). The specific object of learning is also referred to as the direct object of learning while the general object of learning is referred to as the indirect object of learning (Marton et al, 2004).

According to variation theory, careful attention needs to be paid in respect to what is varying and what is invariant in any learning situation in order to understand "what it is possible to learn in that situation and what not" (Marton et al, 2004: 16). There are different patterns of variation as discussed by Marton et al. (2004). These are contrast, generalisation, separation and fusion.

a) Contrast

In order for the learners to discern critical features of a given object of learning, they have to experience something that is not a critical feature of that object of learning so that they compare the different features (Marton et al, 2004). Runesson (2005) argues that a learner is likely to experience something if it is contrasted with "what it is not" (Runesson, 2005: 84), for example, a learner is able to experience *4x+3* as an expression if he is given *4x+3=8* which is not an expression, but an equation. Simply pointing out the features of an algebraic expression is not enough, therefore, something that is not an algebraic expression must be shown to the learners so that they have something to contrast with.

b) Generalisation

In order for learners to experience critical features of a given object of learning, various examples showing the critical features of the object of learning should be given to the learners and this brings out the dimension of the varying aspects of the features of the object of learning (Runesson, 2005), for example, different algebraic expressions such as *x+y, m+4x, y+5x+9, -3x+7g,* and *5x+4y-2x* should be given to the learners so that they can see several examples of how algebraic expressions look like in order to discern critical aspects of algebraic expression.

c) Separation

In order to experience a certain aspect of the object of learning, the aspect should be separated from other aspects by varying that particular aspect and keeping the other aspects invariant (Marton et al, 2004), and this is referred to as controlled variation (Watson and Mason, 2006). For example, in order for the aspect of being able to use various letters in an algebraic expression to be experienced, learners can an expression: x*+y, m+n, z+u,* whereby the structural meaning of the expression *x+y* is invariant but the letters are changing. By giving the learners such expressions, they are able to discern the feature that any letter can be used in an algebraic expression without actually changing the meaning of that expression. Learners discern differences between and within objects through attending to variation (Watson and Mason, 2006). Variation must be controlled because if everything is varying at the same time, nothing maybe discerned., for example, expressions such as x+y, 2x-m+u, n+2m are given for learners to discern the critical feature of being able to use various letters in algebra, many things are changing in these expressions such as the coefficients, the structure and the number of variables in the expressions. Therefore, it may be difficult for the learners to discern the critical feature being put across. In other words, random variation does not help learners to focus their attention on the critical features of the object of learning (Watson and Mason, 2006).

d) Fusion

Learners need to be able to discern different critical features of the object of learning at the same time (Runesson, 2005), and this is referred to as synchronic simultaneity (Marton et al, 2004). Discerning simultaneously is experienced against the background of previous experience (Marton et al, 2004), for example, within an expression such as $4m+x-2z$, the learner should be able to discern that there are three different variables, with different coefficient and one of the variables has a coefficient of one. If the learner is able to discern all the features at the same time, then this is known as synchronic simultaneity.

WHY TEXTBOOK ANALYSIS?

Most learners solve mathematics problems mindlessly and they try to follow the textbook examples without understanding (Mckenzie, 2001). Some learners think that there are easy and hard ways of solving mathematics problems (absence of strategic competence) (Aineamani, 2010). They tend to ask their teachers for easier ways other than what is given in the textbook. There is a gap in some learners' reasoning and communicating mathematically when they move from written to spoken mathematics. Some learners' adaptive reasoning is not well developed because they cannot explain and justify what they have written down (Aineamani, 2010). Therefore, the textbook which the learners uses should have activities that help the learners develop the five strands of mathematics proficiency.

The textbook should be written in such a way that mathematics concepts are well explained to learners even in the absence of the teacher. Teachers should only help learners with mathematics problems which leaners cannot do independently so that learners are given a chance to practice their ideas (Watson, 2009).

ANALYSIS OF THE TEXTBOOKS USING THE THEORIES ABOVE

Conceptual understanding

As discussed earlier, Conceptual understanding refers to the grasp of mathematical ideas (Kilpatrick et al., 2001). In the textbooks analysed, Mathematics vocabulary is explained using multiple representations to communicate conceptual understanding of a word. In the extracts below, the definition of an arithmetic sequence is explained using symbols and words.

Unit 1: Arithmetic sequences

In general we define an **arithmetic sequence** as follows:

$a; \quad a + d; \quad a + 2d; \quad a + 3d; \quad a + 4d; \quad a + 5d; \ldots a + (n - 1)d$

- a is the value of the first term
- d is the **common difference** between the terms, $d = T_2 - T_1 = T_3 - T_2 = T_n - T_{n-1}$
- T_n is the value of the term in position n, so $T_n = a + (n - 1)d$
- n is the position of a term and can only be a positive whole number, also known as a natural number.

arithmetic sequence – a sequence of numbers with a common difference between consecutive terms

Mathematics concepts are modelled using multiple representations to communicate conceptual understanding, for example, in the extract from one of the textbooks below, the textbook emphasises the use of three representations to solve given problems.

Unit 2: Use Venn diagrams, tree diagrams and contingency tables to solve problems

When you have complex, wordy problems involving probabilities, it helps to visualise the data as diagrams. Diagrams make the information easier to understand. Using worked examples, we will discuss how to decide which type of diagram (Venn diagram, tree diagram or contingency table) would be most useful in each situation.

The textbooks which we analysed also use the strategy of concept mapping. The textbooks allow for development of a concept map to help learners discuss the important prerequisite learning concepts. Detailed Teaching guidelines on how the teacher may help learners to develop the concept map are given in the teacher's guide.

Teaching guidelines

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

To understand the theorem, the learners must understand and know that:

- the area of a triangle = $\frac{1}{2}$ base × perpendicular height
- triangles which have the same base and same height are equal in area
- triangles with different bases, but equal heights, are not equal in area.
- the ratio of the areas of triangles with equal heights is equal to ratio of the lengths of their bases.

If two triangles have a common vertex and their bases lie on the same straight line, then the ratio of their areas is equal to the ratio of their bases.

Extract from the Teacher's guide showing how to develop a concept map

Procedural fluency

Procedural fluency refers to the skill in carrying out procedures flexibly, accurately and appropriately and this requires practice (Kilpatrick et al., 2001) and therefore variable examples must be given. The textbooks which we analysed provide students with various activities in order to practice mathematics. Some of the activities in the textbooks are: Various exercises in the learner book, Investigation, projects, mid-year exam papers (In the teacher's guide), term tests, control test book (separate book), question bank (on CD), exam practice papers in both learner book and Teacher's guide, and revision test topics. The textbooks also give learners examples with explanations of how to carry out the procedures appropriately as shown in the extract below.

So, taking the example given above and working in reverse order:

$$x^2 - 2x + y^2 + 10y = -17$$
$$x^2 - 2x + 1 + y^2 + 10y + 25 = -17 + 1 + 25$$

Add 1 and 25 to the right-hand side to balance the fact that those values were added to the left-hand side.

$\left(\frac{1}{2}\text{coefficient of } x\right)^2 = \left(\frac{1}{2} \times -2\right)^2 = 1$

$\left(\frac{1}{2}\text{coefficient of } y\right)^2 = \left(\frac{1}{2} \times 10\right)^2 = 25$

REMEMBER

The constant term of a perfect square trinomial, where the coefficient of the first term is 1, is always the square of $\frac{1}{2}$ the coefficient of the middle term.

Now form both perfect squares, so

$$x^2 - 2x + 1 + y^2 + 10y + 25 = -17 + 1 + 25$$

same sign $\quad \sqrt{1} \quad$ same sign $\quad \sqrt{25}$

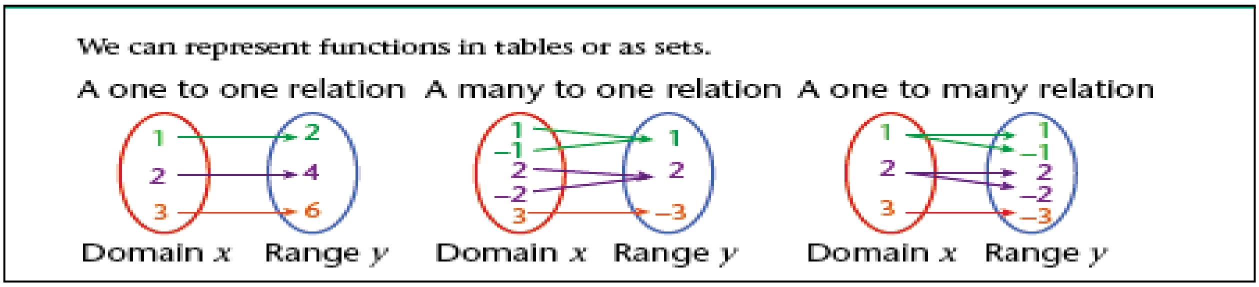
$$\therefore (x - 1)^2 + (y + 5)^2 = 9$$

How the textbook uses Variation theory to develop procedural fluency

Marton et al. (2004) say that they are not advocating for an overall application of variation theory but they believe that variation theory can be used to enable learners to experience the features that are critical for particular learning and also to help learners in developing certain capabilities. There are four patterns of Variation: contrast, generalization, separation and fusion. These four patterns of variation are all catered for in the textbooks which we analysed as shown below.

Contrast

Contract refers to experiencing something that is not a critical feature of that object of learning so that they compare the different features (Marton et al., 2004). In the extract below, a function is represented in such a way that the learner is able to compare the different features such as domain and range of a function.



Generalization

This refers to giving various examples showing the critical features of the object of learning. In the extract below, the textbook gives many examples to capture different aspects.

Determine the radius and centre of the following circles:

3.1 $x^2 + y^2 = 121$

3.2 $3x^2 + 3y^2 - 9 = 0$

3.3 $(x - 1)^2 + (y - 5)^2 = 16$

3.4 $x^2 + (y + 15)^2 = 17$

3.5 $x^2 + 4x + y^2 - 8y = 5$

3.6 $3x^2 + 6x + 3y^2 - 18y + 12 = 0$

In the extract above, different aspects of the equation of a circle have been captured, for example, in 3.1, we see the aspect of both x^2 and y^2 having a coefficient of 1, in 3.2, the coefficient of the variables is not 1. From the activity above, the learners are able to practice different aspects but focusing on the equation of a circle.

Separation

Separation refers to varying a particular aspect and keeping the other aspects invariant. This is also referred to as using controlled variation (Marton et al., 2004). In the extract above, separation has been catered for because the activities given are varying particular aspects of the equation of a circle such as the coefficients of the variables, the radius of the circle and the centre of the circle.

Fusion

Fusion refers to discerning different critical features of the object of learning at the same time and this requires including more than one aspect in an example (Marton et al., 2004). In the extract below, different aspects are included in each of the questions, for example, in question 19, we see surds and binomials, in question 21, we see a fraction, a quadratic equation and binomial. All these questions are set in this way in order to allow the learner to practice different procedures.

19 $f'(x)$ if $f(x) = (\sqrt{x} - 3)(2\sqrt{x} + 3)$	**20** $f'(x)$ if $f(x) = 2x^5 - 3x^4 + \frac{1}{3}x^3 - 4$
21 $g'(x)$ if $g(x) = \frac{3x^2 - 4x - 7}{x + 1}$	**22** $\frac{d}{dx}[(2x - 3)(4x^2 + 6x + 9)]$

Strategic competence

This refers to the ability to formulate, represent, and solve mathematical problems (Kilpatrick et al., 2001). Questions that require learner to come up with a strategy are given, for example, in the extract below, learners are expected to come up with a strategy to solve the problem.

1.3 The second and third terms of a geometric sequence are $\frac{1}{3}$ and 1 respectively. Find the smallest value of n for which the sum of the first n terms exceeds 6.

Adaptive reasoning

This refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick et al., 2001). Questions that require learners to justify and give reasons for their answers are given in the textbooks, as shown in the extract below.

9.3.1	Calculate, with reasons, the size of each angle:	
	a)	$\hat{E}$
	b)	$\hat{F}_2$
	c)	$\hat{B}_3$
	d)	$\hat{C}_1$
	e)	$\hat{C}_2$
9.3.2	Is CD a tangent to the smaller circle? Justify your answer.	
9.3.3	Is EBFD a cyclic quadrilateral? Justify your answer.	

Productive Disposition

This refers to the ability to see mathematics as sensible, useful, and worthwhile. Productive disposition also refers to having belief in yourself and being able to see yourself as one who can do mathematics (Kilpatrick et al., 2001). The textbook provides techniques which teachers can use to help learners develop productive disposition, for example, the text books have sections on inclusive education and integration in the teacher's guide.

Discussion and conclusion

The textbook used in the classroom is very important in determining how learners view mathematics (Mukucha, 2011). The textbook helps in enabling or restricting learners to communicate their mathematics ideas. The textbook should probe learners for higher mathematical reasoning by asking questions that require them to give more explanation and good justifications for their responses in the mathematics classroom. The textbook should also legitimate continuous assessment through exercises, tests, investigations and projects. This will enable learners to practice and hence reflect on their work (Aineamani, 2010). Marton et al. (2004) argue that the learners' focus should be on the direct object of learning, while the teacher's focus should be on both the direct object of learning and the indirect object of learning. Since the textbook is meant to act as the knowledgeable other as the teacher (Vygotsky, 1978), it has to also focus on both the direct and the indirect object of learning. Therefore, the textbook activities should be structured in such a way that the object of learning "comes to the fore of the learners' awareness" (Marton et al., 2004).

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